

AN ARZELÀ-ASCOLI THEOREM FOR THE HAUSDORFF MEASURE OF NONCOMPACTNESS

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ABSTRACT. We generalize the Arzelà-Ascoli theorem in the space of continuous maps on a compact interval with values in Euclidean N -space by providing a quantitative link between the Hausdorff measure of noncompactness in this space and a natural measure of non-uniform equicontinuity. The proof hinges upon a classical result of Jung's on the Chebyshev radius.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Fix $N \in \mathbb{N}_0$ and let $\mathcal{C} = \mathcal{C}([a, b], \mathbb{R}^N)$ be the space of continuous \mathbb{R}^N -valued maps on the compact interval $[a, b]$. Let $|\cdot|$ stand for the Euclidean norm on \mathbb{R}^N and recall that a set $\mathcal{F} \subset \mathcal{C}$ is said to be

- (1) *uniformly bounded* iff there exists a universal constant $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and $x \in [a, b]$,
- (2) *uniformly relatively compact* iff each sequence in \mathcal{F} contains a subsequence converging uniformly to a map in \mathcal{C} ,
- (3) *uniformly equicontinuous* iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ and $x, y \in [a, b]$ with $|x - y| < \delta$.

Denote the collection of uniformly bounded sets in \mathcal{C} as $\mathcal{B}_\mathcal{C}$. In this setting the following theorem is a classic ([L93]).

Theorem 1.1. (*Arzelà-Ascoli*) *For $\mathcal{F} \in \mathcal{B}_\mathcal{C}$ the following are equivalent:*

- (1) \mathcal{F} is *uniformly relatively compact*.
- (2) \mathcal{F} is *uniformly equicontinuous*.

Recall that \mathcal{C} is a Banach space under the supremum norm

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|$$

and that for a set $\mathcal{F} \in \mathcal{B}_\mathcal{C}$ the *Hausdorff measure of noncompactness* ([BG80],[WW96]) is given by

$$\mu_H(\mathcal{F}) = \inf_{\mathcal{F}_0} \sup_{f \in \mathcal{F}} \inf_{g \in \mathcal{F}_0} \|f - g\|_\infty,$$

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the first infimum running through all finite sets \mathcal{F}_0 in \mathcal{C} . It is well known that \mathcal{F} is uniformly relatively compact if and only if $\mu_H(\mathcal{F}) = 0$.

For a set $\mathcal{F} \in \mathcal{B}_\mathcal{C}$ we define the *measure of non-uniform equicontinuity* as

$$\mu_{\text{uec}}(\mathcal{F}) = \inf_{\delta > 0} \sup_{f \in \mathcal{F}} \sup_{|x-y| < \delta} |f(x) - f(y)|,$$

the second supremum running through all $x, y \in [a, b]$ with $|x - y| < \delta$. It is clear that \mathcal{F} is uniformly equicontinuous if and only if $\mu_{\text{uec}}(\mathcal{F}) = 0$. In [BG80] it was shown that μ_{uec} is a measure of noncompactness on the space \mathcal{C} (Theorem 11.2).

Theorem 1.2, our main result, generalizes Theorem 1.1 by linking μ_H and μ_{uec} quantitatively. The proof is deferred to section 3.

Theorem 1.2. (*Arzelà-Ascoli for the Hausdorff measure of noncompactness*) For $\mathcal{F} \in \mathcal{B}_\mathcal{C}$ we have

$$\frac{1}{2}\mu_{\text{uec}}(\mathcal{F}) \leq \mu_H(\mathcal{F}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\text{uec}}(\mathcal{F})$$

In particular, if $N = 1$, then

$$\mu_H(\mathcal{F}) = \frac{1}{2}\mu_{\text{uec}}(\mathcal{F}).$$

2. A PRELIMINARY RESULT OF JUNG'S

For a bounded set $A \subset \mathbb{R}^N$, the *diameter* is given by

$$\text{diam}(A) = \sup_{x, y \in A} |x - y|$$

and the *Chebyshev radius* by

$$r(A) = \inf_{x \in \mathbb{R}^N} \sup_{y \in A} |x - y|.$$

It is well known that for each bounded set $A \subset \mathbb{R}^N$ there exists a unique $x_A \in \mathbb{R}^N$ such that

$$\sup_{y \in A} |x_A - y| = r(A).$$

The point x_A is called the *Chebyshev center* of A . A good exposition of the previous notions in a general normed vector space can be found in [H72], section 33.

Theorem 2.1 provides a relation between the diameter and the Chebyshev radius of a bounded set in \mathbb{R}^N . A beautiful proof can be found in [BW41]. For extensions of the theorem we refer to [A85], [AFS00], [R02] and [NN06].

Theorem 2.1. (*Jung*) For a bounded set $A \subset \mathbb{R}^N$ we have

$$\frac{1}{2}\text{diam}(A) \leq r(A) \leq \left(\frac{N}{2N+2}\right)^{1/2} \text{diam}(A).$$

3. PROOF OF THEOREM 1.2

We first need two simple lemmas on linear interpolation.

For $c_0 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$ we denote the *closed ball with center c_0 and radius r* as $B^*(c_0, r)$.

Lemma 3.1. *Consider $c_1, c_2 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$ and assume that $B^*(c_1, r) \cap B^*(c_2, r) \neq \emptyset$. Let L be the \mathbb{R}^N -valued map on the compact interval $[\alpha, \beta]$ defined by*

$$L(x) = \frac{\beta - x}{\beta - \alpha} c_1 + \frac{x - \alpha}{\beta - \alpha} c_2.$$

Then, for all $x \in [\alpha, \beta]$ and $y \in B^(c_1, r) \cap B^*(c_2, r)$,*

$$|L(x) - y| \leq r.$$

Proof. The calculation

$$\begin{aligned} |L(x) - y| &= \left| \frac{\beta - x}{\beta - \alpha} (c_1 - y) + \frac{x - \alpha}{\beta - \alpha} (c_2 - y) \right| \\ &\leq \frac{\beta - x}{\beta - \alpha} |c_1 - y| + \frac{x - \alpha}{\beta - \alpha} |c_2 - y| \\ &\leq \frac{\beta - x}{\beta - \alpha} r + \frac{x - \alpha}{\beta - \alpha} r \\ &= r \end{aligned}$$

proves the lemma. \square

Lemma 3.2. *Consider $c_1, c_2, y_1, y_2 \in \mathbb{R}^N$ and $\epsilon > 0$ and suppose that $|c_1 - y_1| \leq \epsilon$ and $|c_2 - y_2| \leq \epsilon$. Let L and M be the \mathbb{R}^N -valued maps on the compact interval $[\alpha, \beta]$ defined by*

$$L(x) = \frac{\beta - x}{\beta - \alpha} c_1 + \frac{x - \alpha}{\beta - \alpha} c_2$$

and

$$M(x) = \frac{\beta - x}{\beta - \alpha} y_1 + \frac{x - \alpha}{\beta - \alpha} y_2.$$

Then

$$\|L - M\|_\infty \leq \epsilon.$$

Proof. The calculation

$$\begin{aligned} |L(x) - M(x)| &= \left| \frac{\beta - x}{\beta - \alpha} (c_1 - y_1) + \frac{x - \alpha}{\beta - \alpha} (c_2 - y_2) \right| \\ &\leq \frac{\beta - x}{\beta - \alpha} |c_1 - y_1| + \frac{x - \alpha}{\beta - \alpha} |c_2 - y_2| \\ &\leq \frac{\beta - x}{\beta - \alpha} \epsilon + \frac{x - \alpha}{\beta - \alpha} \epsilon \\ &= \epsilon \end{aligned}$$

proves the lemma. \square

Proof. (of Theorem 1.2) Let $\mathcal{F} \in \mathcal{B}_c$.

We first show that

$$\mu_H(\mathcal{F}) \leq \left(\frac{N}{2N+2} \right)^{1/2} \mu_{\text{uec}}(\mathcal{F}).$$

Fix $\epsilon > 0$. Then, \mathcal{F} being uniformly bounded, we can take a constant $M > 0$ such that

$$\forall f \in \mathcal{F}, \forall x \in [a, b] : |f(x)| \leq M. \quad (1)$$

Pick a finite set $Y \subset \mathbb{R}^N$ for which

$$\forall z \in B^*(0, 3M), \exists y \in Y : |y - z| \leq \epsilon. \quad (2)$$

Now let $0 < \alpha \leq 2M$ be so that $\mu_{\text{uec}}(\mathcal{F}) < \alpha$, i.e. there exists $\delta > 0$ for which

$$\forall f \in \mathcal{F}, \forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq \alpha. \quad (3)$$

Then choose points

$$a = x_0 < x_1 < \dots < x_{2n} < x_{2n+1} = b,$$

put

$$\begin{aligned} I_0 &= [0, x_2[, \\ I_k &=]x_{2k-1}, x_{2k+2}[\text{ if } k \in \{1, \dots, n-1\}, \\ I_n &=]x_{2n-1}, x_{2n+1}] \end{aligned}$$

and assume that we have made this choice such that

$$\forall k \in \{0, \dots, n\} : \text{diam}(I_k) < \delta. \quad (4)$$

Furthermore, for each $(y_0, \dots, y_{2n+1}) \in Y^{2n+2}$, let $L_{(y_0, \dots, y_{2n+1})}$ be the \mathbb{R}^N -valued map on $[a, b]$ defined by

$$L_{(y_0, \dots, y_{2n+1})}(x) = \begin{cases} \frac{x_1-x}{x_1-x_0}y_0 + \frac{x-x_0}{x_1-x_0}y_1 & \text{if } x \in [x_0, x_1] \\ \vdots & \\ \frac{x_{k+1}-x}{x_{k+1}-x_k}y_k + \frac{x-x_k}{x_{k+1}-x_k}y_{k+1} & \text{if } x \in [x_k, x_{k+1}] \\ \vdots & \\ \frac{x_{2n+1}-x}{x_{2n+1}-x_{2n}}y_{2n} + \frac{x-x_{2n}}{x_{2n+1}-x_{2n}}y_{2n+1} & \text{if } x \in [x_{2n}, x_{2n+1}] \end{cases}$$

and put

$$\mathcal{F}_0 = \{L_{(y_0, \dots, y_{2n+1})} \mid (y_0, \dots, y_{2n+1}) \in Y^{2n+2}\}.$$

Then \mathcal{F}_0 is a finite subset of \mathcal{C} . Now fix $f \in \mathcal{F}$ and let $c_{f,k}$ stand for the Chebyshev center of $f(I_k)$ for each $k \in \{0, \dots, n\}$. It follows from (3) and (4) that $\text{diam} f(I_k) \leq \alpha$ and thus, by Theorem 2.1,

$$\forall k \in \{0, \dots, n\} : \sup_{x \in I_k} |c_{f,k} - f(x)| \leq \left(\frac{N}{2N+2} \right)^{1/2} \alpha. \quad (5)$$

Let \tilde{f} be the \mathbb{R}^N -valued map on $[a, b]$ defined by

$$\tilde{f}(x) = \left\{ \begin{array}{ll} C_{f,0} & \text{if } x \in [x_0, x_1] \\ \frac{x_2-x}{x_2-x_1}C_{f,0} + \frac{x-x_1}{x_2-x_1}C_{f,1} & \text{if } x \in [x_1, x_2] \\ C_{f,1} & \text{if } x \in [x_2, x_3] \\ \frac{x_4-x}{x_4-x_3}C_{f,1} + \frac{x-x_3}{x_4-x_3}C_{f,2} & \text{if } x \in [x_3, x_4] \\ \vdots & \\ \frac{x_{2k}-x}{x_{2k}-x_{2k-1}}C_{f,k-1} + \frac{x-x_{2k-1}}{x_{2k}-x_{2k-1}}C_{f,k} & \text{if } x \in [x_{2k-1}, x_{2k}] \\ C_{f,k} & \text{if } x \in [x_{2k}, x_{2k+1}] \\ \frac{x_{2k+2}-x}{x_{2k+2}-x_{2k+1}}C_{f,k} + \frac{x-x_{2k+1}}{x_{2k+2}-x_{2k+1}}C_{f,k+1} & \text{if } x \in [x_{2k+1}, x_{2k+2}] \\ \vdots & \\ \frac{x_{2n-2}-x}{x_{2n-2}-x_{2n-3}}C_{f,n-2} + \frac{x-x_{2n-3}}{x_{2n-2}-x_{2n-3}}C_{f,n-1} & \text{if } x \in [x_{2n-3}, x_{2n-2}] \\ C_{f,n-1} & \text{if } x \in [x_{2n-2}, x_{2n-1}] \\ \frac{x_{2n}-x}{x_{2n}-x_{2n-1}}C_{f,n-1} + \frac{x-x_{2n-1}}{x_{2n}-x_{2n-1}}C_{f,n} & \text{if } x \in [x_{2n-1}, x_{2n}] \\ C_{f,n} & \text{if } x \in [x_{2n}, x_{2n+1}] \end{array} \right. .$$

Then (5) and Lemma 3.1 learn that

$$\|\tilde{f} - f\|_\infty \leq \left(\frac{N}{2N+2} \right)^{1/2} \alpha. \quad (6)$$

Also, it easily follows from (1) and (5) that $\|\tilde{f}\|_\infty \leq 3M$ and thus (2) allows us to choose $(y_0, \dots, y_{2n+1}) \in Y^{2n+2}$ such that

$$\forall k \in \{0, \dots, 2n+1\} : |y_k - \tilde{f}(x_k)| \leq \epsilon. \quad (7)$$

Combining (7) and Lemma 3.2 reveals that

$$\|L_{(y_0, \dots, y_{2n+1})} - \tilde{f}\|_\infty \leq \epsilon. \quad (8)$$

But then we have found $L_{(y_0, \dots, y_{2n+1})}$ in \mathcal{F}_0 for which, by (6) and (8),

$$\|L_{(y_0, \dots, y_{2n+1})} - f\|_\infty \leq \left(\frac{N}{2N+2} \right)^{1/2} \alpha + \epsilon$$

which, by the arbitrariness of ϵ , entails that $\mu_H(\mathcal{F}) \leq \left(\frac{N}{2N+2} \right)^{1/2} \alpha$ and thus, by the arbitrariness of α , the inequality

$$\mu_H(\mathcal{F}) \leq \left(\frac{N}{2N+2} \right)^{1/2} \mu_{\text{uc}}(\mathcal{F})$$

is established.

We now prove that

$$\frac{1}{2} \mu_{\text{uc}}(\mathcal{F}) \leq \mu_H(\mathcal{F}).$$

Let $\alpha > 0$ be so that $\mu_H(\mathcal{F}) < \alpha$. Then there exists a finite set $\mathcal{F}_0 \subset \mathcal{C}$ such that for all $f \in \mathcal{F}$ there exists $g \in \mathcal{F}_0$ for which $\|g - f\|_\infty \leq \alpha$.

Take $\epsilon > 0$. Since \mathcal{F}_0 is uniformly equicontinuous there exists $\delta > 0$ so that

$$\forall g \in \mathcal{F}_0, \forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |g(x) - g(y)| \leq \epsilon. \quad (9)$$

Now, for $f \in \mathcal{F}$, choose $g \in \mathcal{F}_0$ such that

$$\|g - f\|_\infty \leq \alpha. \quad (10)$$

Then, for $x, y \in [a, b]$ with $|x - y| < \delta$, we have, by (9) and (10),

$$|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \leq 2\alpha + \epsilon$$

which, by the arbitrariness of ϵ reveals that $\mu_{\text{uec}}(\mathcal{F}) \leq 2\alpha$ and thus, by the arbitrariness of α , the inequality

$$\frac{1}{2}\mu_{\text{uec}}(\mathcal{F}) \leq \mu_{\text{H}}(\mathcal{F})$$

holds. □

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